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Displacement control crack-growth instability in an elastic-softening material

Part II *Analysis for double-edge-notch configuration*

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The criterion for crack growth instability in an elastic-softening material that is subjected to displacement control loading conditions is examined. A theoretical analysis of the model of a solid containing two symmetrically situated deep cracks and with tensile loading of the remaining ligament, defines the criterion for crack growth instability. The criterion is expressed in terms of the material's softening characteristics and the solid's geometrical parameters. The analysis covers the complete spectrum of material behaviour from the case where the softening zone is very small to the case where instability does not occur until the softening zone traverses the ligament between the crack tips.

1. Introduction

Many materials, for example, ceramics, concretes, cements and fibre-reinforced composites, exhibit a behaviour such that when a pre-cracked solid is progressively loaded, the material fractures at the crack tip and the crack extends; behind the propagating crack tip there is a softening zone of partially fractured material which exerts a restraining stress between the crack faces. This restraining stress is related, via the material softening law, to the relative displacement of the crack faces, and acts until the opening at the trailing edge of the softening zone, i.e. the original crack tip, attains a critical value when the restraining stress falls to zero; the softening zone is then said to be fully developed. Thereafter, the crack continues to extend with a constant opening at the trailing edge of the softening zone. During the last few years, particular attention has been paid to the relation between the crack tip stress intensity, K, as measured at the leading edge of the softening zone, and the crack extension [1–6]. This relation depends on a variety of factors: the geometrical configuration, loading pattern, the softening law and the magnitude of K_{IC}, the fracture toughness of the matrix material, i.e. the crack tip stress intensity needed to fracture the material at the crack tip.

The global response of a cracked solid is another important aspect of the behaviour of an elastic-softening material, and Carpinteri [7] has recently focused attention on this aspect. He considered the situation where $K_{IC} = 0$, i.e., the fracture toughness of the matrix material was presumed to be negligible, and examined the behaviour of a material whose fully developed softening zone size is very large; he investigated the model of an edge-cracked solid that is subjected to bending deformation. By analysing a range of situations where the solid width, length and crack depth were scaled up proportionally, Carpinteri showed that a displacement control crack growth instability of the cusp catastrophe type, i.e. both load and displacement decrease during crack extension, was favoured by large dimensions, and also by a small crack depth–solid width ratio. He referred to experimental results [7] which support the theoretical predictions. In Part I [8] Carpinteri's study of the bend configuration was extended to the case where the softening zone is very small in comparison with other characteristic dimensions of the configuration, i.e. at the opposite end of the spectrum of material behaviour to that considered by Carpinteri [7]. By performing a linear elastic analysis, and assuming that the crack extension condition can be viewed in terms of the stress intensity factor, K, being equal to K_c, a measure of the fracture resistance due to the restraining effects of the softening material, he showed that the criterion for a cusp-type displacement control crack growth instability can be expressed in the form $(a/W) < g(L/W)$ where a = crack depth, W = beam width, L = beam length and $g(L/W)$ is an increasing function of L/W. The criterion is therefore independent of material properties (i.e. K_C) but depends only on geometrical parameters through the ratios a/W and L/W though not on the magnitudes of the dimensions themselves; this is in contrast to Carpinteri's results [7] for a material with a large softening zone, which showed that a cusp in the load–displacement record was favoured by large dimensions.

This paper analyses the model of a solid containing two symmetrically situated deep cracks and with tensile loading of the remaining ligament. With such

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a model, the behaviour of materials having large and small softening zones can be considered within the framework of the same analytical procedure. The analysis defines the condition, expressed in terms of the material's softening zone characteristics and the solid's geometrical parameters, for a displacement control crack growth instability of the cusp catastrophe type. The results are, in general, consistent with those obtained [7, 8] for the bend configuration. The importance of a cusp-type instability stems from the fact that many engineering structures are subjected to displacement control loading, and if there is a cusp-type instability, and though there may be stability on the lower portion of the load–displacement record, according to a static analysis, the energy associated with a sudden load reduction may well lead to catastrophic dynamic failure of the structure.

2. Theoretical analysis

The model (Fig. 1) of a solid of width $2h$, height D and thickness B in the direction of the figure normal is analysed. The solid contains two symmetrically situated deep cracks, and is subjected to an applied relative displacement, Δ, at points along the axis which bisects the ligament, this displacement being associated with a load P. The loading causes the cracks, whose tips are at a distance $2L_0$ apart in the unloaded state, to propagate so that the tips are a distance $2L_*$ apart. There are softening zones of length $(L_0 - L_*)$ at each crack tip, and it is assumed in the first instance that the softening zones are not fully developed. In order to allow for a simple analysis, it is assumed that the restraining stress within a softening zone retains a constant value p_c, this stress being operative until the opening at the trailing edge of a softening zone attains a critical value δ_c. Furthermore, it is assumed that the fracture toughness of the matrix material is zero; i.e. there is finiteness of stress at the crack tips.

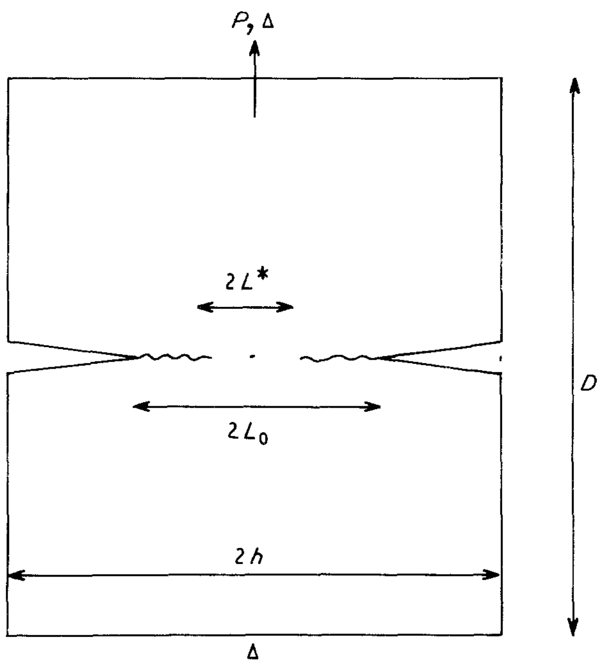

Figure 1 The model analysed in Section 2; the figure shows partially developed softening zones at the crack tips which are initially a distance $2L_0$ apart.

With these simplifications, we can use the results relevant to the Dugdale–Bilby–Cottrell–Swinden (DBCS) [9, 10] representation of plastic deformation at a crack tip.

As indicated above, the initial concern is with regard to the formation of partially developed softening zones at the tips of cracks that are a distance $2L_0$ apart prior to the load application. For this plane strain situation, results from a previous analysis [11] for plastically relaxed cracks give the opening δ_{TIP} at the initial crack tip positions, i.e. at the trailing edges of the partially developed softening zones, as

$$\delta_{TIP} = \frac{4\rho_c L_0}{\pi E_0}[(1+\lambda_0)\ln(1+\lambda_0) + (1-\lambda_0)\ln(1-\lambda_0)] \quad (1)$$

where $E_0 = E/(1-\nu^2)$, E is Young's modulus and ν = Poisson's ratio, and λ_0 is given by the relation

$$\lambda_0 = \frac{P}{2BL_0\rho_c} \quad (2)$$

Furthermore, the relative displacement, Δ, of the loading points is given by the expression

$$\Delta = \left[\frac{D}{2h} + \frac{4}{\pi}\ln\left(\frac{2h}{\pi L_0}\right)\right]\frac{P}{BE_0} + \frac{4\rho_c L_0}{\pi E_0}[2\lambda_0 - (1+\lambda_0)\ln(1+\lambda_0) + (1-\lambda_0)\ln(1-\lambda_0)] \quad (3)$$

the first term on the right-hand side being the displacement when there are no softening zones, and the second term being the displacement due to the softening zones. Equations 1–3 show that as the applied displacement, Δ, increases, then so does the load P, the parameter λ_0 and the displacement δ_{TIP} at the initial crack tips. If δ_c is the critical value of the crack opening, δ_{TIP}, associated with a fully developed softening zone, i.e. the displacement at which the restraining stress falls to zero, Equation 1 shows that the softening zones become fully developed before the zones traverse the ligament between the initial crack tips, i.e. prior to "general softening", if $\pi E_0 \delta_c / 8p_c L_0 \ln 2 < 1$. If this condition is satisfied, then following the full development of the softening zones, the cracks continue to grow with a constant opening δ_c at the trailing edges of the zones. When the trailing edges of these zones are a distance $2L$ apart, Equations 1–3 are replaced by

$$\delta_c = \frac{4p_c L}{\pi E_0}[(1+\lambda)\ln(1+\lambda) + (1-\lambda)\ln(1-\lambda)] \quad (4)$$

$$\lambda = \frac{P}{2BLp_c} \quad (5)$$

$$\Delta = \left[\frac{D}{2h} + \frac{4}{\pi}\ln\left(\frac{2h}{\pi L}\right)\right]\frac{P}{BE_0} + \frac{4p_c L}{\pi E_0}[2\lambda - (1+\lambda)\ln(1+\lambda) + (1-\lambda)\ln(1-\lambda)] \quad (6)$$

with $\lambda < 1$ if general softening has still not occurred. Now Equations 4 and 5, upon differentiation, give, respectively

$$0 = \delta L[(1+\lambda)\ln(1+\lambda) + (1-\lambda)\ln(1-\lambda)] + L\delta\lambda[\ln(1+\lambda) - \ln(1-\lambda)] \quad (7)$$

and

$$\frac{\delta P}{2Bp_c} = \lambda\delta L + L\delta\lambda \quad (8)$$

whereupon

$$\frac{\delta P}{2Bp_c\delta L} = f(\lambda) = -\left[\frac{\ln(1+\lambda) + \ln(1-\lambda)}{\ln(1+\lambda) - \ln(1-\lambda)}\right] \quad (9)$$

Because $f(\lambda)$ is always positive, increasing from zero to unity as λ increases from zero to unity when general softening occurs, Equation 9 shows that the load P decreases during crack extension. Furthermore Equations 5 and 6 show that

$$\frac{\pi E_0\delta\Delta}{4p_c} = L\delta\lambda\left[\frac{\pi D}{4h} + 2\ln\left(\frac{2h}{\pi L}\right)\right] + \lambda\delta L\left[\frac{\pi D}{4h} + 2\ln\left(\frac{2h}{\pi L}\right) - 2\right] + \delta L[2\lambda - (1+\lambda)\ln(1-\lambda) + (1-\lambda)\ln(1-\lambda)] + L\delta\lambda[-\ln(1+\lambda) - \ln(1-\lambda)] \quad (10)$$

and it then follows from Equations 7 and 10 that Δ decreases as L decreases if

$$\left[\frac{\pi D}{4h} + 2\ln\left(\frac{2h}{\pi L}\right)\right] > g(\lambda) = \frac{-4\ln(1-\lambda)\ln(1+\lambda)}{-[\ln(1-\lambda) + \ln(1+\lambda)]} \quad (11)$$

Because P decreases during crack extension, following the full development of softening zones, Equation 11 is the condition for there to be a cusp-type displacement control crack growth instability when the trailing edges of the fully developed softening zones are a distance $2L$ apart.

As a special case, the condition for there to be a cusp-type instability simultaneous with the full development of softening zones at crack tips, which are initially at a distance $2L_0$ apart, is

$$\left[\frac{\pi D}{4h} + 2\ln\left(\frac{2h}{\pi L_0}\right)\right] > g(\lambda_0) = \frac{-4\ln(1-\lambda_0)\ln(1+\lambda_0)}{-[\ln(1-\lambda_0) + \ln(1+\lambda_0)]} \quad (12)$$

with $\lambda_0 = P/2BL_0p_c$. This equation, together with Equation 4 for $\lambda = \lambda_0$, enables us to define the boundary curve separating the regions for which a cusp-type displacement control crack growth instability does and does not occur as soon as a softening zone is fully developed, remembering that we are dealing with the situation where general softening has not occurred, i.e. $\pi E_0\delta_c/8p_cL_0\ln 2 < 1$. Thus with

$$x = \frac{\pi E_0\delta_c}{8p_cL_0\ln 2} \quad (13)$$

and

$$y = \frac{\pi D}{4h} + 2\ln\left(\frac{2h}{\pi L_0}\right) \quad (14)$$

and Equation 4 with λ replaced by λ_0, the appropriate x and y values for the boundary curve can be obtained, and these are given in Table I. The boundary curve AB is shown in Fig. 2, together with the vertical line $x = 1$; when $x > 1$, general softening occurs prior to the full development of softening zones. As a check, we can perform a very small softening zone analysis, by assuming linear elastic behaviour and that the crack tip stress intensity retains a constant value K_* during crack extension. The displacement, Δ, is then given by the expression

$$\Delta = \left[\frac{D}{2h} + \frac{4}{\pi}\ln\left(\frac{2h}{\pi L}\right)\right]\frac{P}{BE_0} \quad (15)$$

with P and K_* being related by the expression

$$K_* = \frac{P}{(\pi L)^{1/2}} \quad (16)$$

With K_* constant, Equations 15 and 16 show that P and Δ both decrease during crack extension, i.e. there is a cusp instability at the onset of crack extension, if

$$\frac{\pi D}{4h} + \frac{2}{\pi}\ln\left(\frac{2h}{\pi L_0}\right) > 4 \quad (17)$$

This result is consistent with the AB curve in Fig. 2, at the position where $x = 0$, $y = 4$, noting that $x \to 0$ implies a vanishingly small softening zone.

The region below the curve AB in Fig. 2 can be partitioned into two regions: (a) a region (II) in which

TABLE I The x, y values defining the boundary curve which separates the regions for which a cusp-type displacement control crack growth instability does and does not occur as soon as a softening zone is fully developed; general softening is assumed not to have occurred

λ_0	x	y
0	0	4.000
0.1	0.007	3.996
0.2	0.029	3.984
0.3	0.066	3.972
0.4	0.119	3.945
0.5	0.188	3.910
0.6	0.278	3.857
0.7	0.390	3.795
0.8	0.531	3.704
0.9	0.714	3.559
1.0	1.000	2.773
		↑ 4 ln 2

Note: $x = \dfrac{\pi E_0\delta_c}{8p_cL_0\ln 2}$, $y = \left[\dfrac{\pi D}{4h} + 2\ln\left(\dfrac{2h}{\pi L_0}\right)\right]$

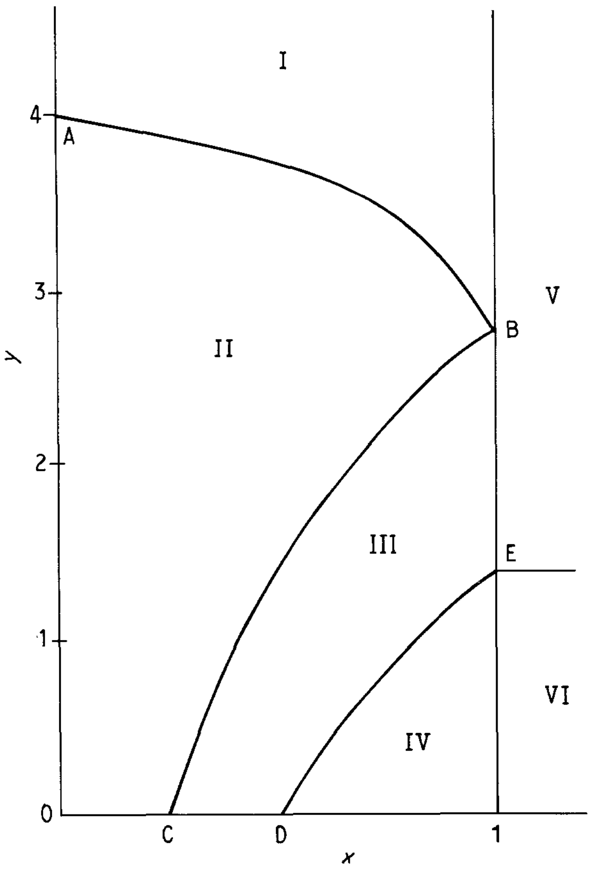

Figure 2 The partitioning of the x–y space into regions with various behaviour patterns; this figure should be viewed in conjunction with Fig. 3. $x = (\pi E_0 \delta_c)/(8 p_c L_0 \ln 2)$, $y = [(\pi D/4h) + 2\ln(2h/\pi L_0)]$. I, Zone fully developed prior to general softening; immediate cusp instability. II, Zone fully developed, with crack extension leading to a cusp instability prior to general softening. III, Zone fully developed, with crack extension leading to general softening and a cusp instability. IV, Zone fully developed, with crack extension leading to general softening and a gradual displacement control instability. V, General softening prior to full development of a softening zone, which is associated with a cusp instability. VI, General softening prior to full development of a softening zone, with crack extension leading to a gradual displacement control instability.

a softening zone is fully developed prior to general softening, but then crack extension proceeds under a decreasing load, P, until there is a cusp-type displacement control instability prior to general softening, and (b) a region (III + IV) in which a softening zone is fully developed prior to general softening, but where crack extension proceeds under a decreasing load until general softening occurs. The boundary curve BC separating these two regions (II and III) is readily shown to be given in terms of x and y values (see Equations 13 and 14) by the relation

$$y = 2\ln 4x \tag{18}$$

If general softening occurs prior to the full development of a softening zone, the load P remains constant at the value $2BL_0 p_c$ (see Equation 5 with $\lambda = 1$), until the zone is fully developed. Thereafter the ligament size decreases from its initial value $2L_0$ and the load decreases; during this stage the non-elastic component, Δ_p, of Δ is given (see the second term in Equation 6, with $\lambda = 1$) by the relation

$$\delta\Delta = \frac{4 p_c \delta L}{\pi E_0}(2 - 2\ln 2) \tag{19}$$

Consequently with $P = 2BLp_c$, and accounting for the elastic contribution (see the first term in Equation 6), it follows that

$$\frac{\pi E_0}{4 p_c}\frac{\delta\Delta}{\delta L} = \frac{\pi D}{4h} + 2\ln\left(\frac{2h}{\pi L}\right) - 2\ln 2 \tag{20}$$

Thus there will eventually be a (gradual) displacement control instability, and this will occur when the ligament size, L, satisfies the condition

$$\frac{\pi D}{4h} + 2\ln\left(\frac{2h}{\pi L}\right) = 2\ln 2 \tag{21}$$

If, however

$$\frac{\pi D}{4h} + 2\ln\left(\frac{2h}{\pi L_0}\right) > 2\ln 2 \tag{22}$$

there is a cusp-type instability immediately a zone is fully developed, without there being any crack extension prior to instability. The boundary (horizontal) line above which the x, y values are such that there is a cusp-type instability as soon as a softening zone is fully developed, is shown in Fig. 2. The existence of this line implies that the region below the curve BC can be partitioned into two sub-regions: (a) a sub-

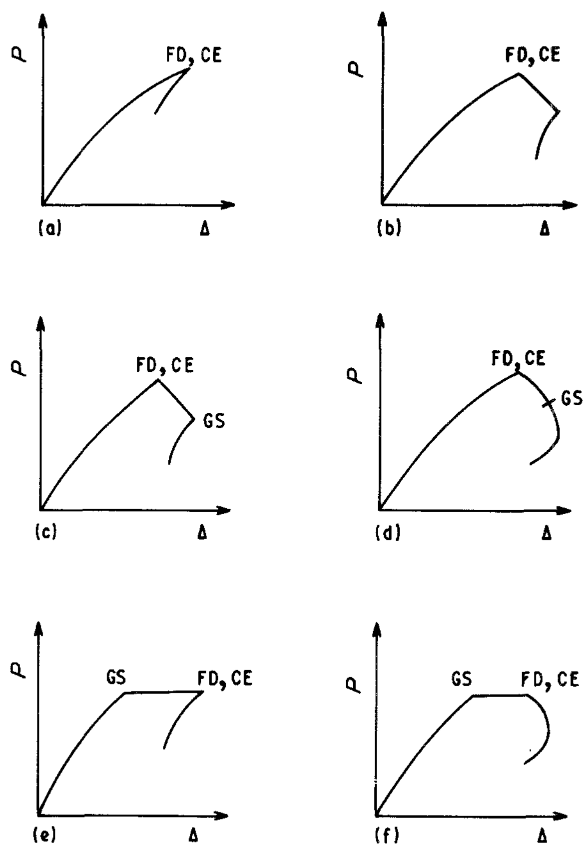

Figure 3 Schematic load (P) – displacement (Δ) records for the various regions in Fig. 2. (a) I, (b) II, (c) III, (d) IV, (e) V and (f) VI. FD, full development of softening zone; CE, crack extension, GS general softening, i.e. softening zone completely traverses the ligament. Crack extension occurs when there is a complete loss of cohesion at the original crack tip.

region (III) where there is crack extension leading to general softening and an immediate cusp-type instability, and (b) a sub-region (IV) where there is crack extension leading to general softening followed by a gradual instability. It is readily shown that the curve DE separating these two sub-regions has the equation

$$y = 2\ln 2x \tag{23}$$

The load–displacement behaviour patterns within the various regions in Fig. 2 are shown schematically in Fig. 3. It should be noted that the displacement Δ, after complete failure ($P = 0$) of the ligament region, is δ_c; the position of this point relative to the various P–Δ curves shown in Fig. 3 depends in large part on the ratio D/h.

3. Discussion

A theoretical analysis of the model of an elastic-softening solid containing two symmetrically situated deep cracks and with tensile loading of the remaining ligament has been presented. To facilitate the analysis it has been assumed that the restraining stress within the softening zone retains a constant value p_c, this stress being effective until the opening at the trailing edge of the softening zone attains a critical value δ_c, when the zone is said to be fully developed. Although the details of the conclusions will be different for a more realistic softening behaviour, e.g. a linear softening law, nevertheless, it is anticipated that the broad general picture will remain the same. The analysis has focused on the condition for there to be a displacement control crack growth instability, and more particularly, a cusp catastrophe instability. Focusing attention on the situation where such an instability occurs prior to general softening, i.e. where softening material completely traverses the ligament, the analysis has defined the instability criterion in terms of the material's softening characteristics, i.e. the parameters p_c and δ_c, and the solid's geometrical parameters, i.e. D, h and L_0. For there to be a cusp catastrophe type of instability prior to general softening, reference to Figs 2 and 3 shows that we should be operating within Regions I and II of the x–y space (Fig. 2). In other words, the following two conditions must be satisfied

$$x = \frac{\pi E_0 \delta_c}{8 p_c L_0 \ln 2} < 1 \tag{24}$$

$$y = \left[\frac{\pi D}{4h} + 2\ln\left(\frac{2h}{\pi L_0}\right)\right] > 2\ln 4x \equiv 2\ln\left[4\left(\frac{\pi E_0 \delta_c}{8 p_c L_0 \ln 2}\right)\right] \tag{25}$$

These relations show that the material and geometrical parameters are coupled in defining the criterion for a cusp-type instability. If the configuration's geometrical parameters are scaled proportionally, Equations 24 and 25 show that instability is favoured by an increase in the solid's dimensions. Furthermore, with prescribed values of D, h and the material parameters, instability is favoured by an increase in L_0, i.e. by a smaller crack depth. These conclusions are consistent with those obtained by Carpinteri [7] for the three-point bend specimen geometry. He analysed the behaviour of one specific material (having a large fully developed softening zone size), but the present analysis leading to Equations 24 and 25 is general in the sense that it embraces the complete spectrum of material behaviour patterns. In this context, these relations show that a cusp-type instability is favoured by a small value of the ratio $E_0\delta_c/L_0 p_c$, i.e. by a small fully developed softening zone size (the fully developed zone size for the reference model of a semi-infinite crack in a remotely loaded infinite solid is $\sim 0.40\, E_0\delta_c/p_c$ [8]). This conclusion is consistent with the result in Part I [8] for the bend specimen geometry for the case where the softening zone size is small, when they are compared with Carpinteri's results; the analysis in Part I was based on linear elastic behaviour of the solid, and the assumption that the crack tip stress intensity retains a constant value K_* during crack extension.

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